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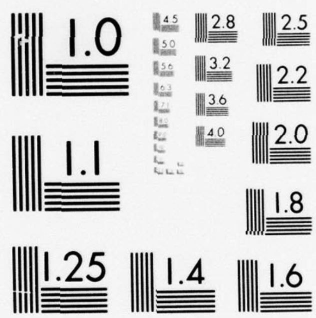


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Computer Information and Control Engineering Program
Department of Industrial and Operations Engineering
Ann Arbor, Michigan 48109

# CHARACTERIZATIONS OF POISSON TRAFFIC STREAMS IN JACKSON QUEUEING NETWORKS

by
Benjamin Melamed

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conjecture to be true for a variety of Jackson networks with single server nodes. Subsequently, a number of characterizations of the equilibrium traffic streams on the arcs of open Jackson networks emerge, whereby stochastic properties of traffic streams are shown to be equivalent to a simple graph-theoretic property of the underlying arcs. These results then help to identify some inherent limitations on the feasibility of equilibrium decompositions of Jackson networks.

## Abstract

The equilibrium behavior of Jackson queueing networks (Poisson arrivals, exponential servers and Bernoulli switches) has recently been investigated in some detail. In particular, it was found that in equilibrium, the traffic processes on the so-called exit arcs of a Jackson network with single server nodes constitute Poisson processes. This result may be viewed as an extension of Burke's Theorem from single queues to networks of queues.

A conjecture made by Burke and others contends that the traffic processes on nonexit arcs cannot be Poisson in equilibrium. This paper proves this conjecture to be true for a variety of Jackson networks with single server nodes. Subsequently, a number of characterizations of the equilibrium traffic streams on the arcs of open Jackson networks emerge, whereby stochastic properties of traffic streams are shown to be equivalent to a simple graph-theoretic property of the underlying arcs. These results then help to identify some inherent limitations on the feasibility of equilibrium decompositions of Jackson networks.



Key Words: Queueing networks, Traffic in queueing networks, Poisson processes, Queueing networks - decomposition, Decomposition of queueing networks

## 1. Introduction.

P.J. Burke [4] and E. Reich [14] showed independently that the departure process from a M/M/s queue in equilibrium is a time homogeneous Poisson process (with same rate as the arrival process). Analogous results have been recently obtained by F.P. Kelly for a large class of queueing networks with exponential servers ([11], [12]), and even for certain ones with general independent servers [1]. These results made elegant use of reversibility, first employed by Reich in [14]. Similar results were attained more directly ([3]; [13] Sec. 4.7) for the class of Jackson queueing network. These networks are termed after R.R.P. Jackson [10] and J.R. Jackson [9]; they are described below.

A *Jackson network* consists of m service stations. Each station i houses a finite number of parallel identical independent servers that provide exponential service times with rate $\sigma_i$. Customers can arrive at a station i either endogenously (from other nodes) or exogenously (from outside the network); in the latter case they arrive from each source according to a Poisson process with intensity $\alpha_i$. When the servers are busy, customers are delayed in a FCFS (first come first serve) waiting line of infinite capacity. On service completion each customer enters a Bernoulli decomposition switch where his next destination is determined according to a multinomial Bernoulli trial. Each customer may then be routed to station j with probability $p_{ij}$, or leave the system altogether with probability $q_i \overset{\Delta}{=} 1 - \sum_{j=1}^{m} p_{ij}$. All exogenous arrival processes, service times and switching decisions are mutually independent.

In this paper, we shall be solely concerned with Jackson networks with single server stations. Such networks are compactly specified by

a quadruple $JN = (M,\alpha,\sigma,P)$ where $M = \{1,2,\ldots,m\}$ is the station set, $\alpha = (\alpha_1,\ldots,\alpha_m)$ is the vector of arrival rates, $\sigma = (\sigma_1,\ldots,\sigma_m)$ is the vector of service rates, and $P = [p_{ij}]$ is a $m \times m$ substochastic matrix which specifies the switching probabilities among stations. The dummy station 0 will denote the network sink, with the convention $p_{00} \triangleq 1$.

Since a network configuration is routinely pictured as a directed graph, we shall find it useful to couch much of the impending discussion in graph-theoretic terms (c.f. [15]). Accordingly, service stations will be termed *nodes*, while the term *arc* will refer to node pairs $(i,j)$ such that $p_{ij} > 0$. A *path* is any sequence of nodes $(k_1,k_2,\ldots,k_n)$ such that $p_{k_\ell k_{\ell+1}} > 0$ for every $1 \le \ell < n$. The *length* of a path $(k_1,k_2,\ldots,k_n)$ is $n-1$, i.e. the number of adjacent arcs specified by the path. If $\alpha_i > 0$ then the network is said to have an *inlet* to node i. Likewise, an arc $(i,0)$ is said to be an *outlet* of the network from node i, if $q_i > 0$.

The *accessibility* relation is defined on $M \cup \{0\}$ as follows: j is accessible from i (notation $i \rightsquigarrow j$) if $i=j$ or there is a path $(i,k_1,\ldots,k_n,j)$. We say that i and j *communicate* if $i \rightsquigarrow j$ and $j \rightsquigarrow i$. The communication relation is an equivalence relation on $M \cup \{0\}$; the equivalence classes of the resultant partition will be termed here *components* of the network.

An elementary classification of nodes and Jackson networks now follows. A node i is *open* if $i \rightsquigarrow 0$; it is *closed* if $i \not\rightsquigarrow 0$. A Jackson network is *open* or *closed* according as all its nodes are open or closed; otherwise, it is *mixed*. A Jackson network is *autonomous* if it has no inlets (see Sec. 4.3 in [13] for more details). A fundamental classification of arcs is given by

Definition 1.1 (cf. [3]; [13] Sec. 4.7)

An arc $(i,j)$ is an exit arc if $j \not\rightsquigarrow i$. Otherwise $(i,j)$ is a nonexit arc. □

Thus, an exit arc is located between distinct components, while nonexit arcs reside within components. It also follows that a nonexit arc (i,j) has a *cycle*, viz. there is a path of the form $(i,j,k_1\ldots,k_{n-1},i)$. Consequently, (i,j) has a cycle of minimal length $L=L(i,j)$; such cycles will be termed minimal cycles (of(i,j)). In particular, when L=1, (i,i) will be called a *feedback* arc.

Let $\{K_{ij}(t)\}_{t\geq 0}$ be the traffic (counting) process on arc (i,j), viz. $K_{ij}(t)$ it the total number of customers that passed on arc (i,j) in the time interval (0,t]. The salient feature of traffic processes on exit arcs (i,j) is that in equilibrium $\{K_{ij}(t)\}_{t\geq 0}$ is a Poisson process. (See [3]; [13] Sec. 4.7). A stronger result will be stated later on in Theorem 2.1.

The study of traffic processes is essential to the study of network decompositions. It was conjectured in [13] (Conjecture 4.7.2) that if (i,j) is a nonexit arc (excluding feedback arcs with $p_{ii}=1$), then $\{K_{ij}(t)\}_{t\geq 0}$ is not a Poisson process. A similar conjecture was previously asserted by P.J. Burke [5], where a heuristic argument to that effect is also outlined. Preliminary results supporting the non-Poisson conjecture were obtained in [13] (see Sec. 4.7).

In this paper, we shall prove a refined version of the aforesaid conjecture as regards feedback arcs in arbitrary Jackson networks; we shall prove it to be true for any open Jackson network in equilibrium. In the latter case, this will yield a Poisson characterization of the traffic processes based on the exit property of the underlying arcs. A number of other characterizations will be seen to emerge from this result; these results suggest general principles that impose inherent limitations on network decompositions. In addition, we shall seek to shed some light on the nature of the deviation from the Poisson, and to point out the interplay between the graph-theoretical aspects and the statistical aspects of Jackson networks.

## 2. Review of Some Known Results

In this section we review some known results that pertain to Jackson networks; the term Jackson networks will henceforth refer to networks with single server nodes.

Let $JN=(M,\alpha,\sigma,P)$ specify such a network. The associated *traffic equation* is

$$(2.1) \qquad \delta = \alpha + \delta P$$

in the unknowns $\delta=(\delta_1,\ldots,\delta_m)$. A solution $\delta\geq 0$ for (2.1) is called a *traffic solution*. An open Jackson network always has a unique traffic solution; an autonomous closed one has infinitely many solutions (see [2], and [13] Sec. 4.4 for more details).

The state process of the network is denoted $\{Q(t)\}_{t\geq 0}$ where $Q(t)=(Q_1(t),\ldots,Q_m(t))$ is the vector of customer totals at each node at time t. $\{Q(t)\}_{t\geq 0}$ is a Markov process ([2]; [13], Theorem 4.2.2) whose state space ranges over all m-tuples $\nu=(n_1,\ldots,n_m)$ of non-negative integers.

For each $\nu=(n_1,\ldots,n_m)$ denote for brevity $P[Q_1(t)=n_1,\ldots,Q_m(t)=n_m] \triangleq P_t(\nu)$, and let $\varepsilon_i$ denote the m-dimensional unit vector with 1 in the i-th coordinate. The birth-and-death equations of $\{Q(t)\}_{t\geq 0}$ are (cf. [9])

$$(2.2) \qquad \frac{\partial}{\partial t} P_t(\nu) =$$

$$\sum_{i=1}^{m} P_t(\nu-\varepsilon_i)\alpha_i$$

$$+ \sum_{j=1}^{m} P_t(\nu+\varepsilon_j)\sigma_j q_j$$

$$+ \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq i}}^{m} P_t(\nu-\varepsilon_i+\varepsilon_j)\sigma_j p_{ji}$$

$$- P_t(\nu)\left(\sum_{i=1}^{m}\alpha_i + \sum_{j=1}^{m}\sigma_j q_j \beta(n_j) + \sum_{i=1}^{m}\sum_{\substack{j=1 \\ j\neq i}}^{m}\sigma_j p_{ji}\beta(n_j)\right),$$

$$\nu=(n_1,\ldots,n_m)\geq 0, \quad t\geq 0,$$

where $\beta(n) = \begin{cases} 0, & \text{if } n=0 \\ 1, & \text{if } n>0 \end{cases}$

A Jackson network is said to be *in equilibrium* if $\{Q(t)\}_{t\geq 0}$ is in steady state; equivalently, $\frac{\partial}{\partial t} P_t(\nu)\equiv 0 \quad \forall\ \nu=(n_1,\ldots,n_m)\geq 0$.

A Jackson network in equilibrium always has a traffic solution $\delta = (E[D_1(t,t+1]],\ldots,E[D_m(t,t+1]])$, where $D_i(t) \triangleq \sum_{j=0}^{m} K_{ij}(t)$ and $D_i(t,u] \triangleq D_i(u)-D_i(t)$ ([2]; [13], Theorem 4.5.4); hence, $\delta$ may depend on the initial condition $P_o(\nu)$, $\nu\geq 0$. The relation between the Traffic Equation and network equilibrium is investigated in detail in Sec. 4.5 of [13]; in particular, if the network is not open, it is necessary in equilibrium for each closed part to be autonomous (ibid.).

For mixed networks in equilibrium, the latter fact enables us to deal separately with each open part and each closed autonomous communicating part.

For open Jackson networks, the equilibrium state distribution was shown by J.R. Jackson [9] to be

(2.3) $$P[Q_1(t)=n_1,\ldots,Q_m(t)=n_m] = \prod_{i=1}^{m}(1-\rho_i)\rho_i^{n_i}$$

under the sufficient condition $\rho_i \triangleq \frac{\delta_i}{\sigma_i} < 1$, $i\in M$. The aforesaid condition is also necessary for equilibrium ([2]; [13] Theorem 4.5.5). Similarly, for autonomous closed networks in equilibrium such that the number of total customers $S_M$ in the network equals n with probability 1, W.J. Gordon and G.F. Newell [8] showed that in equilibrium

(2.4) $$P[Q_1(t)=n_1,\ldots,Q_m(t)=n_m \mid S_M=n] = \frac{1}{g(n)} \prod_{i=1}^{m} \rho_i^{n_i}$$

where $\rho_i \triangleq \frac{\delta_i}{\sigma_i}$ for some traffic solution $\delta$, and $g(n)$ is a normalization constant. Notice that the state space is restricted to $\nu=(n_1,\ldots,n_m)$ such that $\sum_{i=1}^{m} n_i=n$. More generally, if an arbitrary distribution of $S_M$

is known, then in equilibrium

$$(2.5)\qquad P[Q_1(t)=n_1,\ldots,Q_m(t)=n_m] = P[Q_1(t)=n_1,\ldots,Q_m(t)=n_m \mid S_M=\sum_{i=1}^m n_i]\cdot P[S_M=\sum_{i=1}^m n_i].$$

Let (a,b) be an arbitrary arc. Consider the process $\{(Q(t);\ K_{ab}(t))\}_{t\geq 0}$ of the state augmented by the traffic count on arc (a,b). $\{(Q(t);\ K_{ab}(t))\}_{t\geq 0}$ is a Markov process with a countable state space of the form $(\nu;k)\geq 0$ ([3]; [13] Theorem 4.7.1). In writing the corresponding birth-and-death equations,we must distinguish between two cases. Here,the $\varepsilon_i$ are the (m+1)-dimensional unit vectors and $P_t(\nu;k) \overset{\Delta}{=} P[Q_1(t)=n_1,\ldots,Q_m(t)=n_m,K_{ab}(t)=k]$, for any $(\nu;k)=(n_1,\ldots,n_m,k)\geq 0$.

Case 1 (feedback arcs): a=b

$$\begin{aligned}(2.6)\qquad \frac{\partial}{\partial t} P_t(\nu;k) = & \sum_{i=1}^m P_t((\nu;k)-\varepsilon_i)\alpha_i \\ & + \sum_{j=1}^m P_t((\nu;k)+\varepsilon_j)\sigma_j q_j \\ & + \sum_{i=1}^m \sum_{\substack{i=1 \\ j\neq i}}^m P_t((\nu;k)-\varepsilon_i+\varepsilon_j)\sigma_j p_{ji} \\ & + P_t((\nu;k)-\varepsilon_{m+1})\sigma_a p_{aa} \\ & - P_t(\nu;k)[\sum_{i=1}^m \alpha_i + \sum_{j=1}^m \sigma_j q_j \beta(n_j) + \sum_{i=1}^m \sum_{\substack{j=1 \\ j\neq i}}^m \sigma_j p_{ji}\beta(n_j) + \sigma_a p_{aa}\beta(n_a)], \\ & (\nu;k)\geq 0,\ t\geq 0.\end{aligned}$$

Case 2 (nonfeedback arcs): a≠b

$$\begin{aligned}(2.7)\qquad \frac{\partial}{\partial t} P_t(\nu;k) = & \sum_{i=1}^m P_t((\nu;k)-\varepsilon_i)\alpha_i \\ & + \sum_{j=1}^m P_t((\nu;k)+\varepsilon_j)\sigma_j q_j \\ & + \sum_{i=1}^m \sum_{\substack{j=1 \\ j\neq i \\ (j,i)\neq(a,b)}}^m P_t((\nu;k)-\varepsilon_i+\varepsilon_j)\sigma_j p_{ji}\end{aligned}$$

$$+ P_t((\nu;k)-\varepsilon_b+\varepsilon_a-\varepsilon_{m+1})\sigma_a p_{ab}$$

$$- P_t(\nu;k)[\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{m} \sigma_j q_j \beta(n_j) + \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq i}}^{m} \sigma_j p_{ji} \beta(n_j)],$$

$$(\nu;k)\geq 0, \quad t\geq 0.$$

The initial conditions in each case are

(2.8) $$P_o(\nu;k) = \begin{cases} P_o(\nu), & \text{if } k=0 \\ 0 & \text{, otherwise.} \end{cases}$$

Finally, we cite two results pertaining to equilibrium traffic processes.

<u>Theorem 2.1</u>

Let JN=$(M,\alpha,\sigma,P)$ be a Jackson network in equilibrium. Let $(i_1,j_1),\ldots,(i_n,j_n)$ be a set of arcs such that $j_\ell \not\rightarrow i_k$ for all $1\leq \ell \leq n$, $1\leq k\leq n$.

Then the traffic processes $\{K_{i_r j_r}(t)\}_{t\geq o}$, $1\leq r\leq n$, are mutually independent Poisson processes.

<u>Proof</u>

See [3] or [13], Theorem 4.7.3. □

<u>Theorem 2.2</u>

In equilibrium, each component C of the network constitutes itself a Jackson network.

<u>Proof</u>

See [2] or [13], Theorem 4.7.4. □

## 3. Some Implications of a Poisson Traffic Process

Let $\phi_t(z_1,\ldots,z_m) \stackrel{\Delta}{=} \sum_{(n_1,\ldots,n_m)\geq 0} P_t(n_1,\ldots,n_m)\left(\prod_{i=1}^{m} z_i^{n_i}\right)$ be the generating function of the distribution of $Q(t)$. Likewise, the generating function of the distribution of $(Q(t);K_{ab}(t))$ is denoted

$$\phi_t(z_1,\ldots,z_m,y) \stackrel{\Delta}{=} \sum_{(n_1,\ldots,n_m,k)\geq 0} P_t(n_1,\ldots,n_m,k)\left(\prod_{i=1}^{m} z_i^{n_i}\right)y^k.$$

The generating functions above and in the sequel are defined for $t\geq 0$, $|z_i|\leq 1$ $(i\in M)$, and $|y|\leq 1$, unless otherwise specified. We shall, of course, use the convention $0^0 \stackrel{\Delta}{=} 1$. Notice that $\phi_t(z_1,\ldots,1_i,\ldots,z_m,y) \equiv \phi_t(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_m,y)$ where $1_i$ designates the substitution of 1 for $z_i$ in the argument list. This is so, because setting $z_i=1$ results in the generating function of the respective marginal distribution by Abel's Theorem (cf. [9], 1.2). In general, substituting a constant c for $z_i$ in the argument list will be denoted $\phi_t(z_1,\ldots,c_i,\ldots,z_m,y)$. The probability trajectories $P_t(n_1,\ldots,n_m,k)$ have derivatives of every order in t. Moreover, every countable sum thereof is uniformly convergent on each compact interval of $[0,\infty)$. This fact will justify all termwise operations to be performed in the sequel, such as termwise passage to limits, integration, differentiation, etc. (cf. [9], 1.1, 1.7).

We now proceed to write the generating function version of (2.6) and (2.7). By adding and subtracting the term $P_t((\nu;k)-\varepsilon_b+\varepsilon_a)\sigma_a p_{ab}$ in Case 2, we may combine the two cases into one case as follows

$$\begin{aligned}(3.1)\qquad \frac{\partial}{\partial t}\phi_t(z_1,\ldots,z_m,y) &= \sum_{i=1}^{m}\phi_t(z_1,\ldots,z_m,y)\alpha_i(z_i-1) + \\ &+ \sum_{j=1}^{m}(\phi_t(z_1,\ldots,z_m,y)-\phi_t(z_1,\ldots,0_j,\ldots,z_m,y))\sigma_j q_j\left(\frac{1}{z_j}-1\right) \\ &+ \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}(\phi_t(z_1,\ldots,z_m,y)-\phi_t(z_1,\ldots,0_j,\ldots,z_m,y))\sigma_j p_{ji}\left(\frac{z_i}{z_j}-1\right)\end{aligned}$$

$$+ \left(\phi_t(z_1,\ldots,z_m,y)-\phi_t(z_1,\ldots,0_a,\ldots,z_m,y)\right)\sigma_a p_{ab}\left(\frac{z_b y}{z_a}-1\right)$$

$$- \left(\phi_t(z_1,\ldots,z_m,y)-\phi_t(z_1,\ldots,0_a,\ldots,z_m,y)\right)\sigma_a p_{ab}\left(\frac{z_b}{z_a}-1\right).$$

Note that the singularities due to $z_j=0$ in the respective denominators are all removable via continuous extensions.

Next set $z_i=1$, $i\in M$, on both sides of Equation (3.1). We obtain

$$(3.2) \qquad \frac{\partial}{\partial t}\phi_t(y)=(\phi_t(y)-\phi_t(0_a,y))\sigma_a p_{ab}(y-1).$$

In the sequel, $\{B_i(t)\}_{t\geq 0}$ will denote the state indicator process of node i, where

$$B_i(t) = \begin{cases} 0, & \text{if } Q_i(t)=0 \\ 1, & \text{if } Q_i(t)>0. \end{cases}$$

Lemma 3.1

$$(3.3) \qquad E[K_{ab}(t)]=\sigma_a p_{ab}\cdot\int_0^t P[B_a(u)=1]du, \quad t\geq 0.$$

Proof

Equating coefficients in Eq. (3.2) results, after some manipulation, in the system of equations

$$\frac{\partial}{\partial t} P[K_{ab}(t)=k]=$$
$$(P[B_a(t)=1,K_{ab}(t)=k-1]- P[B_a(t)=1,K_{ab}(t)=k])\sigma_a p_{ab},$$
$$k=0,1,\ldots.$$

For each fixed $n\geq 1$, integrate and sum both sides above over $k\geq n$.

In view of (2.8) we get

$$P[K_{ab}(t)\geq n]=\int_0^t P[B_a(u)=1,K_{ab}(u)=n-1]\sigma_a p_{ab}du, \quad n\geq 1.$$

Since $K_{ab}(t)$ is a non-negative integer valued random variable

$$E[K_{ab}(t)]=\sum_{n=1}^{\infty} P[K_{ab}(t)\geq n]=$$

$$\sum_{n=1}^{\infty}\int_{o}^{t} P[B_a(u)=1,K_{ab}(u)=n-1]\sigma_a p_{ab} du=$$

$$\sigma_a p_{ab}\int_{o}^{t} P[B_a(u)=1]du$$

by interchanging summation and integration. □

Corollary 3.1

$E[K_{ab}(t)]=\lambda t$, $t\geq 0$, for some $\lambda\geq 0$ iff $\{B_a(t)\}_{t\geq 0}$ is in steady state. In particular, if $\{K_{ab}(t)\}_{t\geq 0}$ is a time homogenous Poisson process, then $\{B_a(t)\}_{t\geq 0}$ is necessarily in steady state. □

We can now characterize Poisson-distributed traffic processes in Jackson networks.

Theorem 3.1

$K_{ab}(t)$ is Poisson distributed for every $t\geq 0$ iff $B_a(t)$ and $K_{ab}(t)$ are independent for each fixed $t\geq 0$.

Proof

($\Rightarrow$) Suppose that $K_{ab}(t)$ is Poisson distributed. Thus

$$\phi_t(y)=e^{E[K_{ab}(t)](y-1)} ,$$

and consequently from (3.3)

$$\frac{\partial}{\partial t}\phi_t(y)=\phi_t(y)\sigma_a p_{ab} P[B_a(t)=1](y-1).$$

Substituting the above on the left side of (3.2) and simplifying yields

$$\frac{\partial}{\partial t}\phi_t(y) P[B_a(t)=1]=\phi_t(y)-\phi_t(0_a,y), \ |y|<1.$$

Equating coefficients on both sides above results in

$$P[K_{ab}(t)=k]\cdot P[B_a(t)=1]= P[K_{ab}(t)=k]- P[Q_a(t)=0,K_{ab}(t)=k]=$$

$$P[K_{ab}(t)=k,B_a(t)=1], \ k=0,1,\ldots .$$

The requisite independence now follows, since $B_a(t)$ is a zero-one random variable.

($\Leftarrow$) Suppose $B_a(t)$ and $K_{ab}(t)$ are independent for every fixed t. Then $\phi_t(0_a,y)=\phi_t(0_a)\phi_t(y)$, and Equation (3.2) becomes

$$\frac{\partial}{\partial t}\phi_t(y)=(\phi_t(y)-\phi_t(0_a)\phi_t(y))\sigma_a p_{ab}(y-1).$$

This can equivalently be written as

$$\frac{\partial}{\partial t}\phi_t(y)=\phi_t(y)\cdot P[B_a(t)=1]\sigma_a p_{ab}(y-1),$$

and the unique solution is

$$\phi_t(y)=e^{P[B_a(t)=1]\sigma_a p_{ab}(y-1)}.$$

Finally, the above is recognized for each fixed t as the generating function of a Poisson distribution with respective parameters

$P[B_a(t)=1]\sigma_a p_{ab}$. □

Corollary 3.2

If $\{K_{ab}(t)\}_{t\geq 0}$ is a time homogenous Poisson process, then in particular

$$\frac{\partial^n}{\partial t^n}P[Q_a(t)=0,K_{ab}(t)=0]=P[Q_a(t)=0]e^{-E[K_{ab}(1)]t}(-E[K_{ab}(1)])^n$$

□

The following theorem characterizes time homogenous Poisson traffic in Jackson networks.

Theorem 3.2

$\{K_{ab}(t)\}_{t\geq 0}$ is a time homogenous Poisson process iff

a) $\{K_{ab}(t)\}_{t\geq 0}$ is a renewal process

and

b) $\{B_a(t)\}_{t\geq 0}$ is in steady state.

Proof

($\Rightarrow$) If $\{K_{ab}(t)\}_{t\geq 0}$ is a Poisson process, then it is well-known that a) holds, and b) follows from Corollary 3.1.

($\Leftarrow$) Suppose a) and b) hold. From Corollary 3.1 it follows that

$$R(t) \overset{\Delta}{=} E[K_{ab}(t)]=\sigma_a p_{ab}\ P[B_a(0)=1]t$$

where $R(t)$ is the renewal function of $\{K_{ab}(t)\}_{t\geq 0}$. But the only renewal process with $R(t)=\lambda t$ for some $\lambda\geq 0$ is the Poisson process ([6] p.308). □

It should be pointed out that some aspects of the results obtained so far are strongly reminiscent of analogous ones attained in [7] in the context of a single M/G/1 queue.

The reader is reminded, however, that our results apply to any Jackson network with arbitrary topology (open, closed or mixed). With the aid of these results we shall proceed to treat, in the sequel, traffic processes on nonexit arcs. The discussion will be restricted to arcs (a,b) which are traffic-nontrivial in the sense that $E[K_{ab}(t)]\neq 0$, $t\geq 0$, or equivalently $p_{ab}\cdot P[B_a(t)=1]\neq 0$, $t\geq 0$.

## 4. Traffic on Feedback Arcs

Throughout this section, (a,a) will designate a traffic-nontrivial feedback arc.

Before proving the main theorem we need,

**Lemma 4.1**

If $P[B_a(t)=1]=1$ $t\geq 0$, then $p_{aa}=1$.

**Proof**

Setting $z_i=1$ for all $i\in M-\{a\}$ in the generating function version of (2.2) yields (cf. [2]),

$$\frac{\partial}{\partial t}\phi_t(z_a)=$$

$$(\phi_t(z_a)\alpha_a+\sum_{\substack{j=1\\ j\neq a}}^{m}(\phi_t(z_a)-\phi_t(z_a,0_j))\sigma_j p_{ja})(z_a-1)$$

$$+(\phi_t(z_a)-\phi_t(0_a))\sigma_a(1-p_{aa})(\frac{1}{z_a}-1).$$

The inverse transformation of the above to the time domain gives after some manipulation

(4.1)
$$\frac{\partial}{\partial t}P[Q_a(t)=n]=$$

$$(P[Q_a(t)=n-1]-P[Q_a(t)=n])\alpha_a$$

$$+\sum_{\substack{j=1\\ j\neq a}}^{m}(P[Q_a(t)=n-1,Q_j(t)>0]-P[Q_a(t)=n,Q_j(t)>0])\sigma_j p_{ja}$$

$$+(P[Q_a(t)=n+1]-P[Q_a(t)=n]+P[Q_a(t)=0]\delta_{no})\sigma_a(1-p_{aa}),$$

$$n=0,1,2,\ldots$$

where $\delta_{no}$ is Kronecker's delta.

Now, suppose $P[B_a(t)=1]=1$, $t\geq 0$, but $p_{aa}<1$. We show by induction on $n=0,1,\ldots$ that

(4.2) $\quad P[Q_a(t)=k]\equiv 0, \; t\geq 0,$ for all $k\leq n$

which will contradict $P[B_a(t)=1]\equiv 1, \; t\geq 0$. For n=0, (4.2) immediately follows from the supposition.

Assume that (4.2) holds for n>0. In view of the induction hypothesis we obtain from (4.1)

$$0= P[Q_a(t)=n+1]\sigma_a(1-p_{aa})$$

which establishes the induction step since $\sigma_a(1-p_{aa})\neq 0$ by the supposition.

Theorem 4.1

Let $JN=(M,\alpha,\sigma,P)$ specify an arbitrary Jackson network. Then $\{K_{aa}(t)\}_{t\geq 0}$ is a time homogenous Poisson process iff

a) $P[B_a(0)=1]=1$

and

b) $p_{aa}=1$ †

Proof

For a=b send $t\to 0^+$ in Equation (3.1) thus obtaining

$$(4.3) \quad \frac{\partial}{\partial t}\phi_o(z_1,\ldots,z_m,y)=$$
$$\sum_{i=1}^{m}\phi_o(z_1,\ldots,z_m,y)\alpha_i(z_i-1)$$
$$+\sum_{j=1}^{m}(\phi_o(z_1,\ldots,z_m,y)-\phi_o(z_1,\ldots,0_j,\ldots,z_m,y))\sigma_j q_j(\frac{1}{z_j}-1)$$
$$+\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}(\phi_o(z_1,\ldots,z_m,y)-\phi_o(z_1,\ldots,0_j,\ldots,z_m,y))\sigma_j p_{ji}(\frac{z_i}{z_j}-1)$$
$$+(\phi_o(z_1,\ldots,z_m,y)-\phi_o(z_1,\ldots,0_a,\ldots,z_m,y))\sigma_a p_{aa}(y-1).$$

In view of (2.8), the transformed version of the initial condition is

† Notice that in view of Lemma 4.1, a) and b) above are equivalent to $P[B_a(t)=1]\equiv 1, \; t\geq 0$.

(4.4) $\quad \phi_o(z_1,\ldots,z_m,y)\equiv\phi_o(z_1,\ldots,z_m).$

A comparison of (2.7) in Case 1 with (2.2) reveals that in view of (4.4), Equation (4.3) reduces to

(4.5)† $\quad \frac{\partial}{\partial t}\phi_o(z_1,\ldots,z_m,y)=$

$$\frac{\partial}{\partial t}\phi_o(z_1,\ldots,z_m)+(\phi_o(z_1,\ldots,z_m)-\phi_o(z_1,\ldots,0_a,\ldots,z_m))\sigma_a p_{aa}(y-1)$$

Setting in (4.5) $z_i=1$, $i\in M-\{a\}$, yields

(4.6) $\quad \frac{\partial}{\partial t}\phi_o(z_a,y)=\frac{\partial}{\partial t}\phi_o(z_a)+(\phi_o(z_a)-\phi_o(0_a))\sigma_a p_{aa}(y-1)$ .

($\Rightarrow$) Suppose $\{K_{aa}(t)\}$ is a time homogenous Poisson process. On setting $z_a=0$, $y=0$ in (4.6) we observe that

(4.7) $\quad \frac{\partial}{\partial t}P[B_a(0)=0,K_{aa}(0)=0]=\frac{\partial}{\partial t}P[Q_a(0)=0]+0=0$

since $\frac{\partial}{\partial t}P[Q_a(t)=0]\equiv 0$, $t\geq 0$, by Corollary 3.1. But from Corollary 3.2

(4.8) $\quad \frac{\partial}{\partial t}P[B_a(0)=0,K_{aa}(0)=0]=P[Q_a(0)=0](-E[K_{aa}(1)])$ .

From (4.7) and (4.8) we deduce $P[Q_a(0)=0]=0$, as $E[K_{aa}(1)]>0$ by traffic-nontriviality, so that Condition a) is established. Furthermore, by Corollary 3.1

$$P[B_a(t)=1]\equiv 1,\ t\geq 0.$$

Condition b) now follows from Lemma 4.1.

($\Leftarrow$) If Conditions a) and b) hold, then $\{K_{aa}(t)\}_{t\geq 0}$ is a time homogenous Poisson process, since the interdeparture intervals are mutually independent and exponentially distributed random variables with same parameter $\sigma_a$. ||

---

† Observe that (4.5) is valid even when we deal with a singleton autonomous closed network (one with $M=\{a\}$ and $p_{aa}=1$). In this case, all terms on the right side of (4.3) except for the last one are empty and therefore evaluate to zero. However, it can be directly verified that $\frac{\partial}{\partial t}\phi_t(z_a)\equiv 0$, $t\geq 0$, as required.

Theorems 3.1 and 3.2 provide additional information about $\{K_{aa}(t)\}_{t\geq 0}$ in

<u>Corollary 4.1</u>

When $P[Q_a(0)=0]\neq 0$, it follows from Theorem 3.1 that $\{K_{aa}(t)\}_{t\geq 0}$ is not even Poisson distributed. If in addition $\{B_a(t)\}_{t\geq 0}$ is in steady state, it follows from Theorem 3.2 that $\{K_{aa}(t)\}_{t\geq 0}$ is not even a renewal process. □

A close examination of Equations (2.3) and (2.5) reveals that the two assertions above hold, in particular, when the network is in equilibrium.

## 5. Traffic on Nonfeedback Nonexit Arcs

Let (a,b) be a traffic-nontrivial nonfeedback nonexit arc; thus $a \neq b$, and the minimal cycle length (see Sec. 1) satisfies $L(a,b)=L>1$. Our working conjecture is that $\{K_{ab}(t)\}_{t \geq 0}$ can never be a Poisson process. The proof strategy would be analogous to the one in the previous section, namely to demonstrate an inconsistency with Theorem 3.1. This in turn would again enable us to deduce that $\{K_{ab}(t)\}_{t \geq 0}$ is neither Poisson distributed nor is it a renewal process. A more extensive conjecture, consistent with the above, is put forth in the following

Conjecture 5.1

$$\frac{\partial^L}{\partial t^L} P[Q_a(0)=0, K_{ab}(0)=0] =$$

$$P[Q_a(0)=0](-\sigma_a p_{ab}\ P[B_a(0)=1])^L$$

$$+ \sum_{C_{min}(a,b)} d(a,j_1,\ldots,j_L)$$

where the sum is taken over all $(a,j_1,\ldots,j_L)$ in the set $C_{min}(a,b)$ of all minimal cycles of (a,b), and the $d(a,j_1,\ldots,j_L)$ are nonvanishing terms all having the same sign.

Prooving this conjecture would necessitate the computation of $\frac{\partial^L}{\partial t^L} \phi_o(z_1,\ldots,z_m,y)$. We now outline how this may be done.

Let $G^{(m+1)}$ be the set of generating functions (z-transforms) $\gamma(z_1,\ldots,z_m,y)$ of sequences $\{f(\mu)\}_{\mu \geq 0}$, where $\mu=(n_1,\ldots,n_m,k)$ ranges over all (m+1)-tuples of non-negative integers, and such that $\sum_{\mu \geq 0} f(\mu) < \infty$. Next, associate with $JN=(M,\alpha,\sigma,P)$ and arc (a,b) an operator $D: G^{(m+1)} \to G^{(m+1)}$ defined by

$$(5.1) \qquad D[\gamma(z_1,\ldots,z_m,y)] =$$

$$\sum_{i=1}^{m} \gamma(z_1,\ldots,z_m,y)\alpha_i(z_i-1)$$

$$+\sum_{j=1}^{m}(\gamma(z_1,\dots,z_m,y)-\gamma(z_1,\dots,0_j,\dots,z_m,y))\sigma_j q_j(\frac{1}{z_j}-1)$$

$$+\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}(\gamma(z_1,\dots,z_m,y)-\gamma(z_1,\dots,0_j,\dots,z_m,y)\sigma_j p_{ji}(\frac{z_i}{z_j}-1)$$

$$+(\gamma(z_1,\dots,z_m,y)-\gamma(z_1,\dots,0_a,\dots,z_m,y))\sigma_a p_{ab}\frac{z_b}{z_a}(y-1),$$

$$|z_i|\leq 1\ (i\varepsilon M),|y|\leq 1.$$

We make the following observations

1. $\frac{\partial^n}{\partial t^n}\phi_t(z_1,\dots,z_m,y)=D^n[\phi_t(z_1,\dots,z_m,y)]=D[\frac{\partial^{n-1}}{\partial t^{n-1}}\phi_t(z_1,\dots,z_m,y)]$ ;

2. $\lim_{t\to 0^+}D[\phi_t(z_1,\dots,z_m,y)]=D[\lim_{t\to 0^+}\phi_t(z_1,\dots,z_m,y)]$

whence

3. $D^n[\phi_o(z_1,\dots,z_m,y)]=\frac{\partial^n}{\partial t^n}\phi_o(z_1,\dots,z_m,y)$ ;

and finally

4. D is linear viz.

$$D[c_1\gamma_1+c_2\gamma_2]=c_1D[\gamma_1]+c_2D[\gamma_2]$$

for any $\gamma_1,\gamma_2\varepsilon G^{(m+1)}$ and scalars $c_1,c_2$.

Observation 1. reflects the fact that on differentiating both sides of (3.1) n-1 times, $\frac{\partial^n}{\partial t^n}\phi_t(z_1,\dots,z_m,y)$ can be obtained recursively from $\frac{\partial^{n-1}}{\partial t^{n-1}}\phi_t(z_1,\dots,z_m,y)$ by setting the latter on the right side of (3.1) (or equivalently by applying D to $\frac{\partial^{n-1}}{\partial t^{n-1}}\phi_t(z_1,\dots,z_m,y)$). In particular, this is true for t=0 due to the continuity of D in the sense of Observation 2. Thus, the problem of computing $\frac{\partial^L}{\partial t^L}\phi_o(z_1,\dots,z_m,y)$ reduces to the combinatorial problem of calculating $D^L[\phi_o(z_1,\dots,z_m,y]$. A major reduction in the combinatorics involved can be attained by collecting terms – much like in the previous section – and completing them, if necessary, to expressions which vanish under the appropriate conditions. Thus, sending $t\to 0^+$ in (3.1) for $a\neq b$ yields

(5.2) $D[\phi_o(z_1,\ldots,z_m,y)]=$

$$\frac{\partial}{\partial t}\phi_o(z_1,\ldots,z_m)+(\phi_o(z_1,\ldots,z_m,y)-\phi_o(z_1,\ldots,0_a,\ldots,z_m,y))\sigma_a p_{ab}\frac{z_b}{z_a}(y-1)$$

due to (2.8). Notice that $\frac{\partial}{\partial t}\phi_o(z_1,\ldots,z_m)=0$ in equilibrium.

Lemma 5.1

For $\{K_{ab}(t)\}_{t\geq 0}$ to be a Poisson process, it is necessary that for every $n\geq 0$, $\frac{\partial}{\partial t}P[Q_a(0)=n,K_{ab}(0)=0]=\frac{\partial}{\partial t}P[Q_a(0)=n]-P[Q_a(0)=n+1]\sigma_a p_{ab}$.

Proof

Set $z_i=1$ for all $i\neq a$ in (5.2), thus obtaining

$$\frac{\partial}{\partial t}\phi_o(z_a,y)=\frac{\partial}{\partial t}\phi_o(z_a)+(\phi_o(z_a)-\phi_o(0_a))\sigma_a p_{ab}\frac{1}{z_a}(y-1).$$

The Lemma follows by setting above $z_a=0$ and $y=0$. $\square$

In view of Corollaries 3.1 and 3.2 we have

Corollary 5.1

If $\{K_{ab}(t)\}_{t\geq 0}$ is a time homogenous Poisson process, then necessarily

a) $P[Q_a(0)=1]=P[Q_a(0)=0]\cdot P[Q_a(0)>0]$

b) $P[Q_a(0)>1]=P[Q_a(0)>0]\cdot P[Q_a(0)>0]$ $\square$

Corollary 5.1 considerably restricts the feasibility of Poisson traffic processes. An inspection of Equations (2.4) and (2.5) discloses that even in equilibrium a closed network does not, in general, satisfy the necessary conditions of Corollary 5.1. However, from (2.3), an open Jackson network in equilibrium always does. In view of the feasibility of decomposing the latter networks according to Theorem 2.2, we shall now proceed to show that Conjecture 5.1 holds true for open networks in equilibrium.

## 6. Nonfeedback Nonexit Arcs in Equilibrium Open Jackson Networks

In this section we deal with open Jackson networks in equilibrium; recall (see Sec. 1) that the traffic solution $\delta$ satisfies $\delta_i=E[D_i(t,t+1]]$ $t\geq 0$, for any $i\in M$. Throughout this section $(a,b)$, $a\neq b$, designates a traffic-nontrivial nonexit arc. A major simplification will be attained in the impending computations, by taking advantage of the identity

$$(6.1)\qquad \phi_o(z_1,\ldots,z_m,y)-\phi_o(z_1,\ldots,0_j,\ldots,z_m,y)=\phi_o(z_1,\ldots,z_m,y)\rho_j z_j$$

which follows from (2.3) and (2.8). Furthermore, the equilibrium situation brings about a substantial reduction of combinatorics by enabling us to collect and "complete" terms into vanishing expressions. Thus, (5.2) becomes in equilibrium

$$(6.2)\qquad D[\phi_o(z_1,\ldots,z_m,y)]=\phi_o(z_1,\ldots,z_m,y)\delta_a p_{ab} z_b(y-1).$$

A more complex application of the "completion" method will now be exemplified for the computation of $D[\phi_o(z_1,\ldots,z_m,y)(\prod_{k=1}^{n} z_{j_k}^{\ell_k})]$ where $1\leq n\leq m$, $\ell_k\geq 1$, but $a\notin J_n \stackrel{\Delta}{=} \{j_k: 1\leq k\leq n\}$. Denoting $\phi_o(z_1,\ldots,z_m,y)(\prod_{k=1}^{n} z_{j_k}^{\ell_k}) \stackrel{\Delta}{=} \gamma$ and with the aid of (6.2) we compute (for the sake of clarity we shall explicitly list all terms, including the vanishing ones)

$$\begin{aligned}
D[\gamma]= & \\
& \sum_{i\in M} \gamma\alpha_i(z_i-1)\\
& + \sum_{j\in M-J_n} \gamma\rho_j z_j\sigma_j q_j\left(\frac{1}{z_j}-1\right)+\sum_{j\in J_n}(\gamma-0)\sigma_j q_j\left(\frac{1}{z_j}-1\right)\\
& + \sum_{i\in M}\sum_{\substack{j\in M-J_n\\ j\neq i}} \gamma\rho_j z_j\sigma_j p_{ji}\left(\frac{z_i}{z_j}-1\right)+\sum_{i\in M}\sum_{\substack{j\in J_n\\ j\neq i}}(\gamma-0)\sigma_j p_{ji}\left(\frac{z_i}{z_j}-1\right)\\
& + \gamma\rho_a z_a\sigma_a p_{ab}\frac{z_b}{z_a}(y-1)
\end{aligned}$$

Next, we complete the right side of the above to $\frac{\partial}{\partial t}\phi_o(z_1,\ldots,z_m,y)(\prod_{k=1}^{n} z_{j_k}^{\ell_k})=0$ by adding and subtracting terms of the form $\gamma\rho_j z_j\sigma_j q_j(\frac{1}{z_j}-1)$ and $\gamma\rho_j z_j\sigma_j q_j(\frac{z_i}{z_j}-1)$ for $i\varepsilon M$ and $j\varepsilon J_n$. We obtain after some algebraic manipulation

$$
\begin{aligned}
D[\gamma]= \\
&\frac{\partial}{\partial t}\phi_o(z_1,\ldots,z_m,y)(\prod_{k=1}^{n} z_{j_k}^{\ell_k}) \\
&+\sum_{j\varepsilon J_n}\gamma\sigma_j q_j(\frac{1}{z_j}-1)(1-\rho_j z_j) \\
&+\sum_{i\varepsilon M}\sum_{j\varepsilon J_n}\gamma\sigma_j p_{ji}(\frac{z_i}{z_j}-1)(1-\rho_j z_j) \\
&+\gamma\delta_a p_{ab} z_b(y-1)\ .
\end{aligned}
$$

Finally, after collecting terms, the end product of the calculation reduces to

$$
\begin{aligned}
(6.3)\qquad & D[\phi_o(z_1,\ldots,z_m,y)(\prod_{k=1}^{n} z_{j_k}^{\ell_k})]= \\
&\sum_{j\varepsilon J_n}\phi_o(z_1,\ldots,z_m,y)(\prod_{k=1}^{n} z_{j_k}^{\ell_k})\cdot\frac{1}{z_j}(1-\rho_j z_j)(\sigma_j q_j(1-z_j)+\sum_{\substack{i=1\\ i\neq j}}^{m}\sigma_j p_{ji}(z_i-z_j)) \\
&+\phi_o(z_1,\ldots,z_m,y)(\prod_{k=1}^{n} z_{j_k}^{\ell_k})\delta_a p_{ab} z_b(y-1)\ .
\end{aligned}
$$

Let $P_n(b)$ be the set of all paths $(b,j_1,\ldots,j_n)$ that originate from node b and have length $n\geq 1$. Let $V_n(b)\overset{\Delta}{=}\{i\varepsilon M: \exists(b,j_1,\ldots,j_{k-1},i)\varepsilon P_k(b)$ for some $1\leq k\leq n\}$, $n\geq 1$, be the n-neighborhood of b, where for n=0 we define $V_o(b)\overset{\Delta}{=}\{b\}$.

Finally, let $S_n(b)$ be the set of all sums whose terms have the form

$$
(6.4)\qquad \tau(z_1,\ldots,z_m,y)=C(y-1)^r\phi_o(z_1,\ldots,z_m,y)(\prod_{k\varepsilon I_1} z_k^{r_k})\prod_{k\varepsilon I_2}(1-\rho_k z_k)(z_e-z_d)
$$

where

1. C is a scalar

2. $r \geq 0$, $r_k \geq 0$ are integers

3. $I_1, I_2 \subset V_n(b)$

4. $d, e \in V_n(b)$

<u>Lemma 6.1</u>

For any $\tau(z_1,\dots,z_m,y) \in S_n(b)$, $1 \leq n < L$, of the form (6.4) we have $D[\tau(z_1,\dots,z_m,y)] \in S_{n+1}(b)$.

<u>Proof</u>

Expand $(\prod_{k \in I_1} z_k^{r_k}) \prod_{k \in I_2} (1-\rho_k z_k)$ into a (finite) power series in the $z_k$. Thus, $\tau(z_1,\dots,z_m,y)$ is a sum of terms of the form

$$\tau'(z_1,\dots,z_m,y) = C'(y-1)^r \phi_o(z_1,\dots,z_m,y)(\prod_{k \in I} z_k^{\ell_k})(z_e - z_d) \tag{6.5}$$

for some $C'$, $\ell_k \geq 0$, and $I \subset I_1 \cup I_2 \subset V_n(b)$. Recalling that $a \notin V_n(b)$, we shall now compute $D[\tau'(z_1,\dots,z_m,y)]$ with the aid of (6.3); for the sake of brevity we denote $C'(y-1)^r \phi_o(z_1,\dots,z_m,y)(\prod_{k \in I} z_k^{\ell_k}) \triangleq \gamma'$

$$\begin{aligned}
D[\tau'(z_1,\dots,z_m,y)] =\\
&\sum_{j \in I \cup \{e\}} \gamma' z_e \frac{1}{z_j}(1-\rho_j z_j)(\sigma_j q_j (1-z_j) + \sum_{\substack{i=1\\ i \neq j}}^{m} \sigma_j p_{ji}(z_i - z_j))\\
&+ \gamma' z_e \delta_a p_{ab} z_b (y-1)\\
&- \sum_{j \in I \cup \{d\}} \gamma' z_d \frac{1}{z_j}(1-\rho_j z_j)(\sigma_j q_j (1-z_j) + \sum_{\substack{i=1\\ i \neq j}}^{m} \sigma_j p_{ji}(z_i - z_j))\\
&- \gamma' z_d \delta_a p_{ab} z_b (y-1) =\\
&\sum_{j \in I} \gamma' \frac{1}{z_j}(1-\rho_j z_j)(\sigma_j q_j (1-z_j) + \sum_{\substack{i=1\\ i \neq j}}^{m} \sigma_j p_{ji}(z_i - z_j))(z_e - z_d)\\
&+ \sum_{j \in \{e,d\}} \gamma'(1-\rho_j z_j)(\sigma_j q_j (1-z_j) + \sum_{\substack{i=1\\ i \neq j}}^{m} \sigma_j p_{ji}(z_i - z_j))\\
&+ \gamma' \delta_a p_{ab} z_b (y-1)(z_e - z_d).
\end{aligned}$$

An inspection of the right-most side above will verify that $D[\tau'(z_1,\ldots,z_m,y)]\varepsilon S_{n+1}(b)$. The Lemma follows from the linearity of D.

□

We are now in a position to state

Theorem 6.1

Let $JN=(M,\alpha,\sigma,P)$ be an open Jackson network in equilibrium. Then for $1<n\leq L$

(6.6) $$D^n[\phi_o(z_1,\ldots,z_m,y)]=$$

$$\frac{\partial^n}{\partial t^n}\phi_o^*(z_1,\ldots,z_m,y)\cdot z_b^n$$

$$+\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)\sum_{P_{n-1}(b)}\left(\prod_{k=1}^{n-1}(1-\rho_{j_k}z_{j_k})\sigma_{j_k}p_{j_kj_{k+1}}\right)(z_{j_n}-z_{j_{n-1}})$$

$$+R_{n-2}$$

where $\phi_t^*(z_1,\ldots,z_m,y)\overset{\Delta}{=}\phi_o(z_1,\ldots,z_m)e^{\delta_a p_{ab}t(y-1)}$, and $R_{n-2}\varepsilon S_{n-2}(b)$; the sums $\sum_{P_{n-1}(b)}$ here and elsewhere are taken over all $(j_1,\ldots,j_n)\varepsilon P_{n-1}(b)$, unless otherwise specified.

Proof

From (6.2)

(6.7) $$D[\phi_o(z_1,\ldots,z_m,y)]=\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)\cdot z_b\,.$$

The proof will proceed by induction on $n=2,\ldots,L$.

For $n=2$ we compute from (6.7) with the aid of (6.3)

$$D^2[\phi_o(z_1,\ldots,z_m,y)]=D[\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)z_b]=$$

$$\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)(1-\rho_b z_b)(\sigma_b q_b(1-z_b)+\sum_{\substack{i=1\\i\neq b}}^{m}\sigma_b p_{bi}(z_i-z_b))$$

$$+\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)z_b\delta_a p_{ab}z_b(y-1)=$$

$$\frac{\partial^2}{\partial t^2}\,\phi_o^*(z_1,\dots,z_m,y)\,z_b^2$$

$$+\ \frac{\partial}{\partial t}\,\phi_o^*(z_1,\dots,z_m,y)\sum_{P_1(b)}\Big(\prod_{k=1}^{1}(1-\rho_{j_k}z_{j_k})\sigma_{j_k}p_{j_kj_{k+1}}\Big)(z_{j_2}-z_{j_1})$$

$$+\ R_o$$

for some $R_o \varepsilon S_o(b)$, which establishes the induction basis.

Now, assume that $D^n[\phi_o(z_1,\dots,z_m,y)]$ has the requisite form (6.6) for $2 \le n \le L$, and apply D to $D^n[\phi_o(z_1,\dots,z_m,y)]$. From (6.3) the first expression in (6.6) is

$$\text{(6.8)}\qquad D\Big[\frac{\partial^n}{\partial t^n}\,\phi_o^*(z_1,\dots,z_m,y)\,z_b^n\Big]=$$

$$\frac{\partial^{n+1}}{\partial t^{n+1}}\,\phi_o^*(z_1,\dots,z_m,y)\,z_b^{n+1}$$

$$+\ R_1^{(1)}$$

for some $R_1^{(1)} \varepsilon S_1(b) \subset S_{n-1}(b)$.

Consider the second expression in (6.6). By the induction hypohesis, each term $\tau_{(j_1,\dots,j_n)}$ in the sum associated with the path $(j_1,\dots,j_n)\varepsilon P_{n-1}(b)$ belongs to $S_n(b)$. For each fixed $(j_1,\dots,j_n)\varepsilon P_{n-1}(b)$, expand $\prod_{k=1}^{n-1}(1-\rho_{j_k}z_{j_k})$ in the corresponding $\tau_{(j_1,\dots,j_n)}$ as in Lemma 6.1, and set $e \stackrel{\Delta}{=} j_n$ and $d \stackrel{\Delta}{=} j_{n-1}$. Thus for each term $\tau'$ of the form (6.5) such that $\tau_{(j_1,\dots,j_n)} = \sum \tau'$ (we use the results and notation of Lemma 6.1)

$$D[\tau'(z_1,\dots,z_m,y)]=$$

$$\gamma'(1-\rho_{j_n}z_{j_n})\sum_{\substack{i=1\\ i\neq j_n}}^{m}\sigma_{j_n}p_{j_ni}(z_i-z_{j_n})$$

$$+\ R'_{n-1}$$

for some $R'_{n-1}\varepsilon S_{n-1}(b)$. Note that

$\sum_{\tau'}\gamma' = \frac{\partial}{\partial t}\,\phi_o^*(z_1,\dots,z_m,y)\prod_{k=1}^{n-1}(1-\rho_{j_k}z_{j_k})\sigma_j p_{j_kj_{k+1}}$ the sum being taken over all $\tau'$ whose sum comprises $\tau$. Thus, summing over all such $\tau'$, we obtain for each $(j_1,\dots,j_n)\varepsilon P_{n-1}(b)$

$$D[\tau_{(j_1,\ldots,j_n)}(z_1,\ldots,z_m,y)] =$$

$$\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)\Big(\prod_{k=1}^{n-1}(1-\rho_{j_k}z_{j_k})\sigma_{j_k}p_{j_k j_{k+1}}\Big)(1-\rho_{j_n}z_{j_n})\sum_{\substack{i=1\\ i\neq j_n}}^{m}\sigma_{j_n}p_{j_n i}(z_i-z_{j_n})$$

$$+ R''_{n-1} =$$

$$\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)\sum_{\substack{j_{n+1}=1\\ j_{n+1}\neq j_n}}^{m}\Big(\prod_{k=1}^{n}(1-\rho_{j_k}z_{j_k})\sigma_{j_k}p_{j_k j_{k+1}}\Big)(z_{j_{n+1}}-z_{j_n})$$

$$+ R''_{n-1}$$

for some $R''_{n-1}\in\mathcal{S}_{n-1}(b)$.

Summing again the above over all $(j_1,\ldots,j_n)\in\mathcal{P}_{n-1}(b)$ we get

$$D\Big[\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)\sum_{\mathcal{P}_{n-1}(b)}\Big(\prod_{k=1}^{n-1}(1-\rho_{j_k}z_{j_k})\sigma_{j_k}p_{j_k j_{k+1}}\Big)(z_{j_n}-z_{j_{n-1}})\Big]= \tag{6.9}$$

$$\frac{\partial}{\partial t}\phi_o^*(z_1,\ldots,z_m,y)\sum_{\mathcal{P}_n(b)}\Big(\prod_{k=1}^{n}(1-\rho_{j_k}z_{j_k})\sigma_{j_k}p_{j_k j_{k+1}}\Big)(z_{j_{n+1}}-z_{j_n})$$

$$+ R^{(2)}_{n-1}$$

for some $R^{(2)}_{n-1}\in\mathcal{S}_{n-1}(b)$.

Finally, from Lemma 6.1

$$D[R_{n-2}]\in\mathcal{S}_{n-1}(b). \tag{6.10}$$

The induction step follows from (6.8), (6.9) and (6.10) by linearity of D. □

The main result now follows.

Theorem 6.2

Let JN=(M,α,σ,P) be an open Jackson network in equilibrium. Then $\{K_{ab}(t)\}_{t\geq 0}$ is not a Poisson process.

Proof

For n=L, set in (6.6) $z_i=1$ for all $i\neq a$. Since $a\in\mathcal{V}_{L-1}(b)-\mathcal{V}_{L-2}(b)$,

$R_{L-2}$ vanishes along with all terms corresponding to paths $(j_1,\dots,j_L)\in P_{L-1}(b)$ with $j_L\neq a$, leaving only terms that correspond to all minimal cycles of $(a,b)$. Thus,

(6.11) $$\frac{\partial^L}{\partial t^L}\phi_o(z_a,y)=$$

$$\phi_o(z_a,y)(\delta_a p_{ab}(y-1))^L$$

$$+\sum_{C_{min}(a,b)}\left(\prod_{k=1}^{L-1}(1-\rho_{j_k})\sigma_{j_k}p_{j_k j_{k+1}}\right)(z_a-1)\ ,$$

where $\sum_{C_{min}(a,b)}$ is taken over all $(a,j_1,\dots,j_L)\in C_{min}(a,b)$.

On setting $z_a=0, y=0$ above, we obtain

(6.12) $$\frac{\partial^L}{\partial t^L}P[B_a(0)=0,K_{ab}(0)=0]=$$

$$P[B_a(0)=0,K_{ab}(0)=0](-\delta_a p_{ab})^L$$

$$-\sum_{C_{min}(a,b)}\left(\prod_{k=1}^{L-1}(1-\rho_{j_k})\sigma_{j_k}p_{j_k j_{k+1}}\right)$$

Now, if $\{K_{ab}(t)\}_{t\geq 0}$ is a Poisson process, then from Corollary 3.2

(6.13) $$\frac{\partial^L}{\partial t^L}P[B_a(0)=0,K_{ab}(0)=0]=$$

$$\lim_{t\to 0^+}P[B_a(0)=0]e^{-\delta_a p_{ab}t}(-\delta_a p_{ab})^L=$$

$$P[B_a(0)=0,K_{ab}(0)=0](-\delta_a p_{ab})^L\ .$$

But for each $(a,j_1,\dots,j_L)\in C_{min}(a,b)$ we have $\prod_{k=1}^{L-1}(1-\rho_{j_k})\sigma_{j_k}p_{j_k j_{k+1}}>0$.

As $C_{min}(a,b)\neq\phi$ by our assumption, (6.12) contradicts (6.13) whence the Theorem follows.

||

## 7. Conclusions

The intuitive essence of the preceding results may be loosely described as follows. The traffic process on arc (a,b) is statistically dependent on all state indicators of those nodes which it affects (all nodes accessible from node b). These dependencies radiate about node b in a sort of a ripple effect; they manifest themselves for nodes which are n-distant from b through $\frac{\partial^{n+1}}{\partial t^{n+1}} \phi_o(z_1,\ldots,z_m,y)$. The existence of a minimal cycle for (a,b) enables these dependencies to propagate to node a, thereby leading to a contradiction to Theorem 3.1.

As a consequence of Theorem 6.2, one may summarize the foregoing discussion concerning traffic processes in equilibrium open Jackson network, as follows.

**Conclusion 7.1**

Let $JN=(M,\alpha,\sigma,P)$ be an open Jackson network in equilibrium, and let (a,b) be a traffic-nontrivial arc. Then the following are equivalent statements:

a) (a,b) is an exit arc.

b) $\{K_{ab}(t)\}_{t\geq 0}$ is a Poisson process.

c) $\{K_{ab}(t)\}_{t\geq 0}$ is a renewal process.

d) For each $t\geq 0$, $K_{ab}(t)$ is Poisson distributed with parameter $\delta_a p_{ab} t$.

e) For each $t\geq 0$, $B_a(t)$ and $K_{ab}(t)$ are independent random variables.

f) There is $N\subset M$, $a\in N$, such that for each $t\geq 0$, $K_{ab}(t)$ is independent of $\{Q_i(t): i\in N\}$; in fact, $N=\{i: i \not\to a\}$. ||

Another conclusion reflecting on Theorem 2.2 tells us about the limitations inherent in equilibrium decompositions of open Jackson networks.

Conclusion 7.2

The decomposition in Theorem 2.2 of an open Jackson network in equilibrium into Jackson network components is maximal in the sense that no refinement of the underlying partition will yield Jackson subnetworks (in equilibrium). ||

Next, let us extend the notion of an exit arc to any superposition of traffic processes on arcs emanating from the same node, as follows: the superposition above is said to have the *exit property*, if all its constituent arcs are exit arcs; otherwise, it has the *nonexit property*. In the latter case, the minimal cycle associated with it is the smallest over all nonexit arcs in the superposition. We point out that all the foregoing results will still hold *mutatis mutandis* for such superpositions.

Going back to Conjecture 5.1 and comparing it with Corollary 3.2, we see that the sum $d \overset{\Delta}{=} \left| \sum_{C_{min}(a,b)} d(a,j_1,\ldots,j_L) \right|$ has a useful interpretation; it may be construed as a heuristic deviation of $\{K_{ab}(t)\}_{t\geq 0}$ from the Poisson.† Moreover, each term in the sum may be interpreted as the deviation due to the respective minimal cycle of (a,b).

In particular, for open Jackson networks in equilibrium, this deviation takes on the form

$$(7.1)\qquad d = \begin{cases} P[Q_a(t)=0]\delta_a p_{aa}, \text{ if } L=1 \\ \sum_{C_{min}(a,b)} \left( \prod_{k=1}^{L-1} P[Q_{j_k}(t)=0]\delta_{j_k} p_{j_k j_{k+1}} \right), \text{ if } L>1 \\ 0, \text{ if } L \text{ does not exist (i.e. } C_{min}(a,b) = \phi) \end{cases}$$

owing to (4.7), (4.8) and (6.11).

† To support this view we point out that in view of b) in Conclusion 7.1, $\{K_{ab}(t)\}_{t\geq 0}$ is a Poisson process iff $C_{min}(a,b)=\phi$, and consequently iff $d=0$.

Equation (7.1) provides some clues to the behavior of the deviation as a function of the network's structural parameters. A number of quick observations now follow:

1. The deviation magnitude is affected by network connectivity in the sense that it increases as the number of minimal cycles increases.
2. The deviation component due to a minimal cycle is affected by distance, in the sense that it decreases as the minimal cycle length L increases, provided the service efforts $\rho_{j_k} p_{j_k j_{k+1}}$ along arcs $(j_k, j_{k+1})$ are less than one.
3. The deviation component due to a minimal cycle is affected by the cycle's "strength". It decreases as the cycle gets "weaker", in the sense that the switching probabilities $p_{j_k j_{k+1}}$ decrease in magnitude.
4. The deviation component due to minimal cycle are affected by the work load. It decreases as the nodes tracing it get busier, i.e. as the traffic intensities $\rho_{j_k}$ approach one.

As a matter of fact on approaching saturation, the respective deviations tend to 0.† Consequently, under heavy traffic conditions, over all paths in $C_{min}(a,b)$, $\{K_{ab}(t)\}_{t\geq 0}$ is approximately a Poisson process.

Finally, we mention that analogous versions of the results contained in this paper can be conceivably derived for a generalized notion of traffic processes in Markovian systems. The feasibility of such generalizations is currently under investigation.

† If the point $\infty$ is added to the state space of each node, then the equilibrium notion will include the case $P[Q_a(t)=\infty]=1$ (saturation). Notice that a saturated node is indistinguishable from the environment source because all customer streams departing that node are mutually independent Poisson processes which are also independent of the arrivals, services and switching decisions in all other nodes.

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